Why My Bike Is Like A Truck When It Comes To Climbing Hills?
or
The Magic of Dimensional Analysis

As told by Phil to Robin Ford, October 2010

Robin: Remind us how this question of bikes and trucks came up.

Phil: I wanted to know why loaded trucks slow down more on hills than cars do. So with some friends I worked on predicting vehicle performance. Just for fun.

Robin: How did you go about it?

Phil: We started with speed up an incline. A hermit helped us get equations for tractive effort and for resistance to motion.

Robin: Tractive effort is the drive force at the wheels from the engine, I suppose: resistance to motion is aerodynamic drag, rolling resistance — that sort of thing.

Phil: Right. At steady speed effort and resistance are equal. So, to find vehicle speed, plot both of them against speed on the same graph, and speed is where the two curves cross.

Robin: I can see that. How did you work out the effect of hills?

Phil: We just repeated the calculations for a range of inclines. Look at Figure 1.

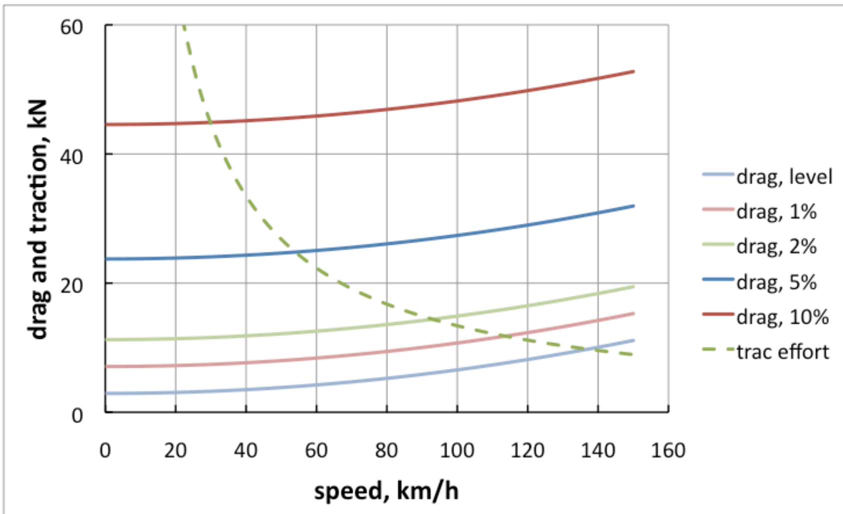

Fig 1 drag and tractive effort vs speed for various inclines

Robin: How do you do that?

Phil: I'll give you the equations later. We made it easier by assuming that the gearing can be adjusted to provide maximum power from the engine at all times.

Robin: Maybe like a continuously variable gearbox?

Phil: Maybe. Anyway, the balance speed is where a drag curve meets the traction curve. Speed drops back as the grade increases, of course. Figure 1 shows you how.

Robin. Reasonable. What next?

Phil: We wanted a good way to compare vehicles — trucks, cars, bikes.

Robin: How could that work? The range of forces and speeds would be too big. You couldn't show them all clearly on one chart?

Phil: Exactly! Luckily a waiter at *Bellissimo* suggested that we divide the forces by vehicle weight to make them 'dimensionless'. That compressed the range so we only needed one chart.

Robin: And what did that show you?

Phil: I saw that the curves were similar for a bike and a truck, but quite different for a car. Look at Figure 2.

Robin: None of those look the same to me.

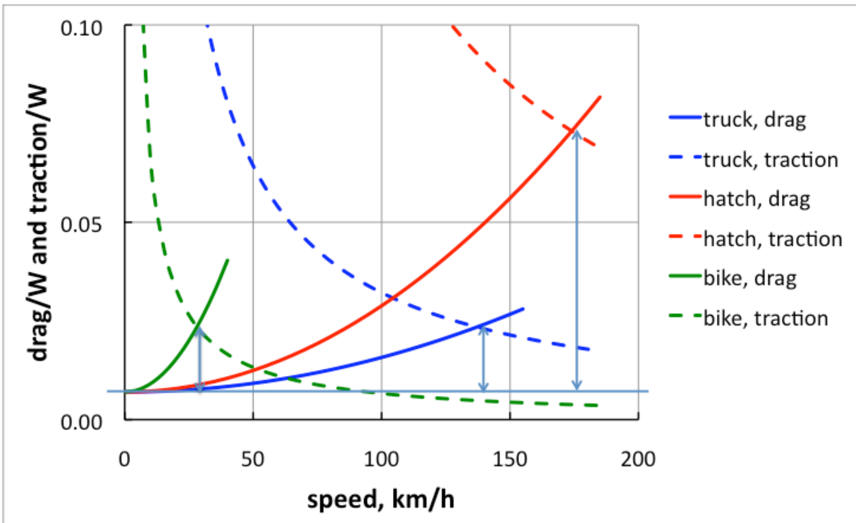

Fig 2 drag and tractive effort vs speed, with aerodynamic components shown by arrows

Phil: Just look at the dimensionless aerodynamic drag at maximum speed. It's similar for the bike and the truck, but different for the car.

Robin: You might be right.

Phil: My friend Pat wasn't impressed either, so that's why I added those arrows on the figure. They show the aero parts of the drag.

Robin: Can you explain it to me now? How did you start?

Phil: With that helpful waiter at the coffee shop — the one who suggested dividing the forces by weights to make them dimensionless. I went back for more help.

Robin: What happened?

Phil: The waiter was busy, and just said, "Check out *Dimensional Analysis*." So I found a book about it, and searched the web.

Robin: And?

Phil: Well, you can reduce the number of variables you need for analysis by combining them into dimensionless groups.

Robin: Really?

Phil: First you work out the quantities you need for describing what you're looking at. Next you count how many of them there are. Then you subtract the number of basic dimensions and, *voila*, you've got the number of dimensionless groups you need. Finally, you build the groups you want — you have options.

Robin: Might be clearer with an example.

Phil: Okay. Look at Figure 3. It lists what we'll need for our problem.

Let's start with the equations for drag. It looks like we'll need seven quantities. But we can keep some quantities together and reduce that number.

Robin: Which ones?

Phil: This is tricky. C_D is a quantity but it doesn't have dimensions, so we can combine A and C_D to give AC_D. And, α and C_R are quantities too but don't have dimensions either, and because they both only appear in combination with W, we can combine α and C_R to give $(\alpha + C_R)$. We are down to five quantities for total drag: AC_D, ρ, v, W, and $(\alpha + C_R)$.

Robin: What about tractive effort?

Phil: We'll keep those quantities separate. The only new one is the power P, so we now have six quantities altogether.

Robin: You've listed the equations and the quantities in Figure 3.

<u>Equations</u>

$$D = \frac{1}{2}AC_D\rho v^2 + W\alpha + WC_R$$

$T = P/v$

<u>Quantities</u>

Tractive effort. T

Total Drag D

Steady speed v

Power at the drive wheels P

Vehicle weight (N) W

Air density ρ

Drag area AC_D

Rolling coefficient (ratio) C_R

Gradient (ratio) α

Fig 3 equations and quantities

You haven't listed the dimensions. They'll be the three most basic ones — length, mass and time.

Phil: Right. That's 6 quantities and 3 dimensions, so we'll need 6 - 3 = 3 dimensionless groups to describe the results.

Robin: And you've shown the groups you came up with in the next list — Figure 4. Why did you choose that set?

Phil: I wanted to plot steady speed versus gradient, so I found one group for speed and one for gradient. And I kept speed and gradient out of the other group.

Robin: That third group looks like aerodynamic drag divided by weight; but speed's missing.

vW/P (includes speed)

$\alpha + C_R$ (specifies gradient and rolling resistance)

$AC_D\rho\,(P/W)^2/2W$ (no v; no α, no C_R)

Fig 4 dimensionless groups chosen

Phil: Yes. I did that on purpose. I used P/W instead — P/W and speed have the same dimensions. Actually, P/W is the speed of the vehicle if it could pull itself vertically upwards with gearing that gives maximum power at that speed.

Robin: That's complicated. I'll just look at the results in Figure 5. Please explain.

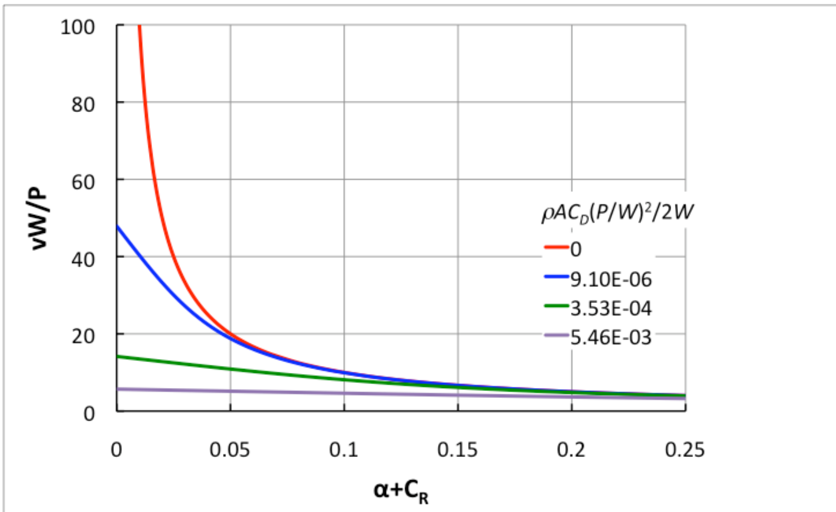

Fig 5 dimensionless speed vs (slope + rolling coefficient) for various $\rho AC_D(P/W)^2/2W$

Phil: Well, first look how simple it is with just three groups.

Robin: Obviously.

Phil: The vertical axis is dimensionless speed, vW/P, and the horizontal axis is $\alpha + C_R$.

Robin: I see. When $\alpha + C_R = 0$ it's going down a hill where gravity balances rolling resistance.

Phil: Quite.

Robin: What about the top curve? It looks like a reciprocal. Does it go to infinity at $\alpha + C_R = 0$?

Phil: Right; it *is* a reciprocal. It's the dimensionless speed for no air resistance — and at $\alpha + C_R = 0$ gravity balances rolling resistance so there is no resistance at all and the steady speed is infinite. You can never have that of course.

Robin: What about the other curves?

Phil: Look at this table.

Vehicle	$\boldsymbol{\rho A C_D (P/W)^2/2W}$
hatchback	$3.53\text{x}10^{-4}$
truck	$9.1\text{x}10^{-6}$
bicycle	$9.1\text{x}10^{-6}$

The small hatchback car has $\rho A C_D(P/W)^2/2W = 3.53\text{x}10^{-4}$, and the truck and bike share $\rho A C_D(P/W)^2/2W = 9.1\text{x}10^{-6}$. The value for truck and bike is much smaller; it's more like the no air resistance case.

Robin: You're telling me that my bike is like a truck because they both have the same low value of $\rho A C_D(P/W)^2/2W$?

Phil: You've got it.

Robin: But what does it mean?

Phil: The formula for $\rho A C_D(P/W)^2/2W$ says:

"Find the air resistance at the speed the vehicle can pull itself vertically upwards (suitably geared), then divide by the weight." The higher the value, the more air resistance matters compared to weight and the sooner the curve peels off from the no-air-resistance curve.

Robin: Like the bottom curve. What's that one for?

Phil: That's a high-performance car — power to mass ratio of 145kW/tonne. It can go so fast up ordinary grades that air resistance always dominates and the top speed doesn't depend so much on gradients.

Robin: So, for bikes and loaded trucks gradient is what counts most. Both slow down more on hills than cars.

I'd like to see how the figures work out. Let me try.

I'll take the bike and truck. The graph has, in round figures:

$vW/P = 19$

$\alpha + C_R = 0.05.$

So. How do I find the actual speeds for the truck and the bike?

Phil: You need to know the ratio P/W at the drive wheels in each case — in Watts/Newton.

Ok. I get it. What should I take for P/W?

Phil: For the truck t*ry:*

$P/W = 0.9$ W/N

and for the bike:

0.18 W/N

Robin: Right. I'll combine them with $vW/P = 19$:

$v_{truck} = 19 \text{x} 0.9 \text{x} 3.6$

$= 62$ km/h

$v_{bike} = 19 \text{x} 0.18 \text{x} 3.6$

12 km/h.

Phil: You remembered the 3.6 that converts from metres per second to kilometres per hour.

Robin: Now for the hatchback.

Phil: I'll do that.

From the graph I get:

vW/P is 11 or so

$\alpha + C_R = 0.05$,

and with a typical figure for P/W :

$P/W = 3.5$

I get:

$v_{hatch} = 11 \text{x} 3.5 \text{x} 3.6$

$= 139$ km/h,

which is a lot more than the bike or the truck.

Robin: P/W makes a big difference.

Phil: Yes. Well, the waiter at *Bellissimo* told us that. But that's not enough. You need to know $\rho AC_D(P/W)^2/2W$ too.

Robin: So, we've found the speeds at $\alpha + C_R. = 0.05$.

But what actual grade does that correspond to?

Phil: Right. We'll need a typical value for coefficient of rolling resistance, say:

$C_R = 0.007$,

Then, for

$\alpha + C_R. = 0.05$,

we get

$\alpha = 0.043$

or a gradient of about

1 in 23

— what you might find on a major highway.

Robin: On the Sydney to Canberra freeway it's 1:10 in places.

I'll try it for

$\alpha + C_R = 0.11$.

For the hatch, I get:

$vW/P = 7.6$,

v is 96 km/h,

For the truck and the bike vW/P is the same:

$vW/P = 9$

But the speeds are different of course.

For the truck:

v is 29 km/h

For the bike:

v is 5.8 km/h.

It'll be clearer in a table:

Vehicle	**vW/P at $\alpha + C_R = 0.05$**	**max speed at $\alpha + C_R = 0.05$ km/h**	**vW/P at $\alpha + C_R = 0.11$**	**max speed at $\alpha + C_R = 0.11$ km/h**	**% of speed at $\alpha + C_R = 0.05$**
hatchback	6	139	7.6	96	69
truck	6.5	62	9	29	47
bicycle	6.5	12	9	5.8	48

Phil: Good. Look. You can see that the truck and bike both lose a bigger fraction of their maximum speed capability when they hit a hill.

Robin: So, the hatchback can hold its speed better. We knew that! What if we analyse a steeper hill and compare it with a level road.

Phil: I've already done that. Here's a table. Neat, isn't it?

Vehicle	**max speed, level road** $\alpha + C_R = 0.007$	**max speed on a steeper hill** $\alpha + C_R = 0.2$	**% of speed on level road**
hatchback	173	60	35
truck	139	16	9
bicycle	28	3.2	11

Robin: Yes. What about head winds?

Phil: Let's leave that to another time.